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COSINE FAMILIES AND DAMPED SECOND ORDER DIFFERENTIAL EQUATIONS.(U)
AUG 81 J H LIGHTBOURNE, S M RANKIN DAAG29-80-C-0041
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ABSTRACT

Consider the abstract differential equation

$$(1)\quad u''(t) + 2Bu'(t) = Au(t) + F(u(t)),\ t \in \mathbb{R},\ u(0) = x,\quad u'(0) = y$$

where A and B are densely defined linear operators and F is possibly nonlinear and unbounded. Assuming that $A + B^2$ generates a cosine family $C(t)$ and $-B$ generates a group $T(t)$, there is a variation of constants formula for (1); namely

$$(2)\quad u(t) = T(t)[C(t)x + S(t)(Bx + y) + \int_0^t T(t-s)S(t-s)F(u(s))\,ds,$$

where $S(t)$ is the sine family associated with $C(t)$. The motivating examples include $w_{tt} + 2b(x)w_t = w_{xx} + f(w,w_x,w_t)$ and $w_{tt} + 2w_{tx} = w_{xx} + f(w,w_x,w_t)$, for $0 < x < \pi$, $t \in \mathbb{R}$, $w(x,0) = h(x)$, $w_t(x,0) = g(x)$, and various boundary conditions. We examine the existence of mild solutions and the asymptotic behavior when there is a damping effect introduced by the $2Bu'(t)$ term.

AMS(MOS) Subject Classification: 34G20, 35L15
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SIGNIFICANCE AND EXPLANATION

A variation of constants formula is given for certain second order differential equations in a Banach space. The abstract results obtained can be applied to a class of damped semilinear hyperbolic partial differential equations; in particular, the existence and asymptotic behavior of solutions of such equations is examined.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

COSINE FAMILIES AND DAMPED SECOND ORDER DIFFERENTIAL EQUATIONS

James H. Lightbourne, III
Samuel M. Rankin, III

1. INTRODUCTION. Let X be a Banach space and A and B be linear operators on X with domains $D(A)$ and $D(B)$ respectively. F will denote a nonlinear, possibly unbounded map on X. We consider the abstract differential equation:

$$(1.1) \qquad u''(t) + 2Bu'(t) = Au(t) + F(u(t)), \qquad t \in \mathbb{R}$$

$$u(0) = x, \quad u'(0) = y$$

Essentially under the assumption that $A + B^2$ generates a cosine family $C(t)$, $t \in R$, of linear operators on X and that $-B$ generates a group $T(t)$, we will establish existence for (1.1) and examine the asymptotic behavior of (1.1) when there is a damping effect introduced by the term $2Bu'(t)$. We will actually consider "mild" solutions of (1.1); i.e., solutions of the variation of constants equation:

$$(1.2) \qquad u(t) = T(t)[C(t)x + S(t)(Bx + y)] + \int_0^t T(t - s)S(t - s)F(u(s))ds,$$

where $S(t)$ is the sine family associated with $C(t)$.

Two situations to which the abstract theory applies are indicated by the following examples.

$$(1.3) \qquad w_{tt}(x,t) + 2b(x)w_t(x,t) = w_{xx}(x,t) + f(w(x,t), w_x(x,t)),$$
$$t \in \mathbb{R}, \quad 0 < x < \pi$$

$$w(x,0) = h(x), \quad w_t(x,0) = g(x), \qquad 0 \le x \le \pi$$

$$w(0,t) = w(\pi,t) = 0, \qquad t \in \mathbb{R}$$

where $b: [0,\pi] \to \mathbb{R}$ is continuous. Asymptotic behavior for (1.3)

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and $f \equiv 0$ has been considered by Rauch [3]. The second illustrative example is

$$w_{tt}(x,t) + 2w_{xt}(x,t) = w_{xx}(x,t) + f(w(x,t), w_x(x,t)), \quad t \in \mathbb{R}, \quad 0 < x < \pi \tag{1.4}$$

$$w(x,0) = h(x), \quad w_t(x,0) = g(x), \qquad 0 \le x \le \pi$$

$$w(0,t) = w(\pi,t), \quad w_x(0,t) = w_x(\pi,t), \qquad t \in \mathbb{R}$$

The preliminaries in section 2 include the known properties of cosine families that we will use and the assumptions on A and B which will be made throughout the paper. Also, in section 2 we establish the relationship between equations (1.1) and (1.2). In section 3 we give existence criteria for equation (1.2) and some global properties of solutions are given in section 4. The examples are discussed in section 5.

2. PRELIMINARIES. Let X be a Banach space with norm $|\cdot|$.

DEFINITION. A one-parameter family $\{C(t): t \in \mathbb{R}\}$ of bounded linear operators on X is called a strongly continuous cosine family provided

(i) $C(0) = I$, the identity on X;

(ii) $C(s + t) + C(s - t) = 2C(s)C(t)$, for all $s,t \in \mathbb{R}$; and

(iii) for each $x \in X$, $C(\cdot)x: \mathbb{R} \to X$ is continuous.

Associated with $C(t)$ is the sine family $\{S(t): t \in \mathbb{R}\}$ defined by $S(t)x = \int_0^t C(s)x\,ds$ for $x \in X$. the infinitesimal generator of $C(t)$ is the linear operator $G: D(G) \to X$ defined by $Gx = C''(0)x$ where

$$D(G) = \{x \in X: C(\cdot)x: \mathbb{R} \to X \text{ is twice continuously differentiable}\}.$$

We also refer to the set E defined by

$$E = \{x \in X: C(\cdot)x: \mathbb{R} \to X \text{ is continuously differentiable}\}.$$

The proof of the following proposition as well as a more complete discussion of cosine families may be found in Travis and Webb [4].

PROPOSITION 2.1. *Let* $\{C(t): t \in \mathbb{R}\}$ *be a strongly continuous cosine family of bounded linear operators on* X *with generator* G. *The following properties hold:*

(i) G *is a closed operator on* X *with domain* $D(G)$ *dense in* X;

(ii) *if* $x \in X$, *then* $S(t)x \in E$ *and* $S'(t)x = C(t)x$;

(iii) *if* $x \in E$, *then* $S(t)x \in D(G)$ *and* $S''(t)x = GS(t)x$;

(iv) *if* $x \in E$, *then* $C'(t)x = GS(t)x$;

(v) *if* $x \in D(G)$, *then* $S(t)x \in D(G)$ *and* $GS(t)x = S(t)Gx$;

(vi) *if* $x \in D(G)$, *then* $C(t)x \in D(G)$ *and* $C''(t)x = GC(t)x = C(t)Gx$;

(vii) $C(t + s) - C(t - s) = 2GS(t)S(s)$, *for all* $s,t \in \mathbb{R}$;

(viii) $S(s + t) = S(s)C(t) + S(t)C(s)$, *for all* $s,t \in \mathbb{R}$;

(ix) $C(t)$, $S(s)$, $C(s)$, $S(t)$ *commute for* $s,t \in \mathbb{R}$;

(x) *there exist constants* $K \geq 1$ *and* $\omega \geq 0$ *such that* $|C(t)| \leq Ke^{\omega|t|}$ *and* $|S(t) - S(\hat{t})| \leq K|\int_{\hat{t}}^{t} e^{\omega|s|}\,ds|$ *for all* $t,\hat{t} \in \mathbb{R}$.

Throughout this paper we will make the following suppositions on A and B. Recall that $\{T(t): t \in \mathbb{R}\}$ is said to be a strongly continuous

group of linear operators on X provided $T(0) = I$, $T(t + s) = T(t)T(s)$ for all $t,s \in \mathbb{R}$, and for each $x \in X$, $T(\cdot)x: \mathbb{R} \to X$ is continuous.

(2.1) A and B are densely defined linear operators on X with domains $D(A) \subseteq D(B)$.

(2.2) $G = A + B^2$ generates a strongly continuous cosine family $\{C(t): t \in \mathbb{R}\}$.

(2.3) $-B$ generates a strongly continuous group $\{T(t): t \in \mathbb{R}\}$ of linear operators on X.

We will also refer to the regularity conditions:

(2.4) $D(G) = D(A + B^2) \subseteq D(A)$.

(2.5) $T(t): D(A) \to D(A)$.

(2.6) $E \subseteq D(B)$ and if $\{S(t): t \in \mathbb{R}\}$ is the sine family associated with $C(t)$, then $t \to BS(t)x$ is continuous for each $x \in X$ and if $x \in D(B)$ then $S(t)Bx = BS(t)x$.

(2.7) If $x \in D(B)$, then $C(t)x \in D(B)$ and $C(t)Bx = BC(t)x$.

(2.8) For $x \in X$, $\int_r^s T(u)S(u)x\,du \in D(A)$ and

$$A\int_r^s T(u)S(u)x\,du = T(s)C(s)x - T(r)C(r)x + BT(s)S(s)x - BT(r)S(r)x.$$

The authors do not know if (2.8) is a consequence of (2.1) - (2.7); however, it is observed in section 5 that the examples satisfy (2.1) - (2.8).

PROPOSITION 2.2 (Travis and Webb [5]). *Suppose* P *is a closed linear operator on* X *such that*

(i) $S(t) \in D(P)$ *for all* $t \in \mathbb{R}$ *and* $x \in X$; *and*

(ii) for each $x \in X$, *the map* $t \to PS(t)x$ *is continuous.*

Then there exists $M \geq 1$ *and* $\omega^* \geq \omega$ *such that* $|PS(t)| \leq Me^{\omega^*|t|}$ *for all* $t \in \mathbb{R}$, *where* ω *is given in Proposition 2.1(x).*

REMARK. Assuming condition (2.6), there exists $M \geq 1$ and $\omega^* \geq \omega$ such that $|BS(t)| \leq Me^{\omega^*|t|}$.

The following proposition justifies referring to a solution of equation (1.2) as a mild solution of (1.1).

PROPOSITION 2.3. *Suppose* (2.1) - (2.8) *hold and* $g: \mathbb{R} \to X$ *is continuous. If* $g: \mathbb{R} \to X$ *is continuously differentiable,* $x \in D(G)$ *with* $Bx \in E$, $y \in E$, *and* u *satisfies*

$$(2.9) \quad u(t) = T(t)[C(t)x + S(t)(Bx + y)] + \int_0^t T(t-s)S(t-s)g(s)\,ds,$$

then $u(t) \in D(A)$, $u'(t) \in D(B)$ *for all* $t \in \mathbb{R}$, u *is twice continuously differentiable, and* u *satisfies*

$$(2.10) \qquad \begin{aligned} &u''(t) + 2Bu'(t) = Au(t) + g(t) \\ &u(0) = x, \quad u'(0) = y. \end{aligned}$$

Conversely, if u *is twice continuously differentiable,* $u(t) \in D(A)$ *and* $u'(t) \in D(B)$ *for* $t \in \mathbb{R}$, *and* u *satisfies* (2.10), *then* u *satisfies* (2.9).

Proof. To show that a solution of equation (2.9) satisfies (2.10), we first define

$$v(t) = \int_0^t T(t-s)S(t-s)g(s)\,ds.$$

Then

$$v(t) = \int_0^t T(t-s)S(t-s)g(0)\,ds + \int_0^t \int_u^t T(t-s)S(t-s)g'(u)\,ds\,du$$

$$= \int_0^t T(t-s)S(t-s)g(0)\,ds + \int_0^t \int_0^{t-u} T(s)S(s)g'(u)\,ds\,du$$

Using condition (2.8), we have $v(t) \in D(A)$ and

$$Av(t) = T(t)C(t)g(0) - g(0) + BT(t)S(t)g(0)$$

$$+ \int_0^t [T(t-u)C(t-u)g'(u) - g'(u) + BT(t-u)S(t-u)g'(u)]\,du$$

$$= T(t)C(t)g(0) - g(0) + BT(t)S(t)g(0)$$

$$+ \int_0^t [T(t-u)C(t-u)g'(u) + BT(t-u)S(t-u)g'(u)]\,du - g(t).$$

Also,

$$v'(t) = T(t)S(t)g(0) + \int_0^t T(s)S(s)g'(t-s)\,ds$$

$$= T(t)S(t)g(0) + \int_0^t T(t-s)S(t-s)g'(s)\,ds$$

and

$$v''(t) = T(t)C(t)g(0) - BT(t)S(t)g(0)$$

$$+ \int_0^t [BT(t-s)S(t-s)g'(s) + T(t-s)C(t-s)g'(s)]\,ds$$

$$= Av(t) + g(t)$$

Defining $V_H(t) = T(t)[C(t)x + S(t)(Bx + y)]$ and using a straightforward computation, one can establish that

$$v_H''(t) + 2Bv_H'(t) = Av_H(t).$$

Noting that $u(t) = v_H(t) + v(t)$, it follows that u satisfies (2.10).

To establish the converse statement, observe that

$$\frac{d}{ds} T(t-s)S(t-s)u'(s) = T(t-s)[-C(t-s)u(s) + S(t-s)u'(s)]$$
$$+ BT(t-s)S(t-s)u'(s)$$
$$= -T(t-s)C(t-s)u'(s)$$
$$+ T(t-s)S(t-s)[Au(s) - 2Bu'(s) + g(s)]$$
$$+ BT(t-s)S(t-s)u'(s)$$

and

$$\frac{d}{ds} T(t-s)C(t-s)u(s) = T(t-s)[-(A + B^2)S(t-s)u(s) + C(t-s)u'(s)]$$
$$+ BT(t-s)C(t-s)u(s)$$

Integrating we obtain

$$-T(t)S(t)u(0) = \int_0^t [-T(t-s)C(t-s)u'(s) + T(t-s)S(t-s)Au(s)$$
$$- T(t-s)S(t-s)Bu'(s) + T(t-s)S(t-s)g(s)]\, ds$$

and

$$u(t) - T(t)C(t)u(0) = \int_0^t [-T(t-s)AS(t-s)u(s) - T(t-s)B^2S(t-s)u(s)$$
$$+ T(t-s)C(t-s)u'(s) + BT(t-s)C(t-s)u(s)]\, ds$$

Addition of the two formulas yields

$$u(t) - T(t)S(t)u'(0) - T(t)C(t)u(0)$$

$$= \int_0^t T(t-s)S(t-s)g(s)\,ds$$

$$+ \int_0^t [BT(t-s)C(t-s)u(s) - T(t-s)S(t-s)Bu'(s)$$

$$- T(t-s)B^2S(t-s)u(s)]\,ds$$

$$= \int_0^t T(t-s)S(t-s)g(s)\,ds - \int_0^t \frac{d}{ds}[T(t-s)S(t-s)Bu(s)]\,ds$$

$$= \int_0^t T(t-s)S(t-s)g(s)\,ds + T(t)S(t)Bu(0)$$

and it is seen that u satisfies (2.9).

3. EXISTENCE. *In this section we establish the existence of solutions* to equation (1.2) under various assumptions on the cosine family $C(t)$ generated by $G = A + B^2$ and the nonlinear function F.

PROPOSITION 3.1 (Fattorini [1]). *If* G *is the generator of a strongly continuous cosine family then there exists a translation* $G_c \equiv G - c^2 I$ *of* G *such that*

(i) G_c^{-1} *exists as a bounded operator on* X *and*

(ii) for $0 \le \alpha \le 1$ *the fractional powers* $(-G_c)^{\alpha}$ *exist as closed, densely defined operators with*
$D(G) \subset D((-G_c)^{\alpha_1}) \subset D((-G_c)^{\alpha_2})$ *for* $0 \le \alpha_2 < \alpha_1 \le 1$.

The existence of $(-G_c)^{-1}$ implies that $(-G_c)^{-\alpha}$ exists as a bounded linear operator on X and consequently $D((-G_c)^{\alpha})$ becomes a Banach space X_α with norm $|x|_\alpha = |(-G_c)^{\alpha}x|$. Also in [1], it was shown that if $X = \mathcal{L}^p$,

$1 < p < \infty$ then $E \subset D((-G_c)^{1/2})$ and for each $x \in X$, $(-G_c)^{1/2}S(\cdot)x\colon \mathbb{R} \to X$ is continuous. For general Banach space X, Rankin [2] showed that $\tilde{E} \subset D((-G_c)^{\alpha})$ for all $0 \le \alpha < 1/2$. We shall make the following assumptions:

(3.1) There exists $0 < \lambda < 1$ such that $E \subset D((-G_c)^{\lambda})$ and $(-G_c)^{\lambda}S(\cdot)x\colon \mathbb{R} \to X$ is continuous for $x \in X$.

(3.2) If $0 \le \alpha \le 1$ and $x \in D((-G_c)^{\alpha})$, then $T(t)x \in D((-G_c)^{\alpha})$ with $(-G_c)^{\alpha}T(t)x = T(t)(-G_c)^{\alpha}x$ for all $t \in \mathbb{R}$.

REMARK. If condition (3.1) holds, then by Proposition 2.2 there exists $M_\lambda \ge 1$ and $\omega_\lambda \ge \omega$ such that $|(-G_c)^{\lambda}S(t)| \le M_\lambda e^{\omega_\lambda |t|}$ for all $t \in \mathbb{R}$.

THEOREM 3.1. *In addition to* (2.1) - (2.3), (3.1), *and* (3.2), *suppose* $D \subset X_\lambda$ *is open. If* $F\colon D \to X$ *satisfies* $|F(x_1) - F(x_2)| \le L|x_1 - x_2|_\lambda$ *for some* $L > 0$ *and all* $x_1, x_2 \in D$, *then for each* $x \in D \cap D(B)$ *and* $y \in X$ *there exists* $a > 0$ *and a unique continuous function* $u\colon [-a,a] \to X_\lambda$ *such that* u *satisfies* (1.2).

Proof. The proof employs the contraction mapping principle. Choose $\delta > 0$ and $N > 0$ such that if

$$W(x,\delta) = \{z \in X : |z - x|_\lambda < \delta\}$$

then $W(x,\delta) \subset D$ and $|F(z)| \le N$ for $z \in W(x,\delta)$. Choose $a > 0$ such that for $t \in [-a,a]$

$$|T(t)C(t)x - x|_\lambda + |T(t)S(t)(Bx + y)|_\lambda \le \frac{\delta}{2},$$

$$N\int_{-a}^{a} |T(s)(-G_c)^{\lambda}S(s)|\, ds < \frac{\delta}{2}, \quad \text{and}$$

$$L\int_{-a}^{a} |T(s)(-G_c)^{\lambda}S(s)|\, ds < 1.$$

Define

$$K = \{v \in \mathcal{C}([-a,a];\ X_\lambda):\ \sup_{-a\le t\le a} |v(t) - x|_\lambda \le \delta\}$$

and the map $H: K \to \mathcal{C}([-a,a];\ X_\lambda)$ by

$$[Hv](t) = T(t)[C(t)x + S(t)(Bx + y)] + \int_0^t T(t-s)S(t-s)F(v(s))\,ds$$

The choice of δ and a implies that $H: K \to K$. Furthermore, for $v_1, v_2 \in K$

$$|Hv_1(t) - Hv_2(t)|_\lambda \le \int_0^t |T(t-s)(-G_c)^\lambda S(t-s)[F(v_1(s)) - F(v_2(s))]|\,ds$$

$$\le L\int_0^t |T(t-s)(-G_c)^\lambda S(t-s)|\ |v_1(s) - v_2(s)|_\lambda\,ds$$

and by the choice of a we have that H satisfies the hypothesis of the contraction mapping principle. The assertions follow.

THEOREM 3.2. *In addition to assumptions* (2.1) - (2.5), (3.1), *and* (3.2), *suppose* $(-G_c)^{-1}$ *is compact. Let* $D \subset X_\lambda$ *be open and* $0 < \beta$. *If* $F: D \to X_\beta$ *is continuous, then for each* $x \in D \cap D(B)$ *and* $y \in X$ *there exist* $a > 0$ *and a continuous function* $u: [-a,a] \to X_\lambda$ *satisfying* (1.2).

Proof. Let $\delta > 0$ and $N > 0$ be such that if

$$W(x,\delta) = \{z \in X : |z - x|_\lambda < \delta\}$$

then $W(x,\delta) \subset D$ and $|F(z)|_\beta \le N$ for $z \in w(x,\delta)$. Choose $a > 0$ such that

$$|T(t)C(t)x - x|_\lambda + |T(t)S(t)(Bx + y)|_\lambda \le \frac{\delta}{2}$$

and

$$N\int_{-a}^{a} |T(s)(-G_c)^\lambda S(s)|\,ds < \frac{\delta}{2}\,.$$

Define the set K and map H as in the proof of Theorem 3.1. As in Theorem 3.1, the choice of δ and a implies that $H: K \to K$. For $v_1, v_2 \in K$,

$$|Hv_1(t) - Hv_2(t)|_\lambda \le \int_0^t |T(t-s)(-G_c)^\lambda S(t-s)|\;|F(v_1(s)) - F(v_2(s))|\,ds$$

and the continuity of H follows from the continuity of $F: D \to X_\beta$. To show that $\{Hv: v \in K\}$ is an equicontinuous family in $\mathcal{C}([-a,a]; X_\lambda)$, we observe that if $(G_c)^{-1}$ is compact then $(-G_c)^{-\alpha}$ is compact for $0 < \alpha < 1$ (see Travis and Webb [6]). For $-a \le t_1 < t_2 \le a$ and $v \in K$

$$\begin{aligned}
&|Hv(t_1) - Hv(t_2)|_\lambda \\
&\le |T(t_2)C(t_2)x - T(t_1)C(t_1)x|_\lambda + |T(t_2)S(t_2)(Bx+y) - T(t_1)S(t_1)(Bx+y)|_\lambda \\
&\quad + \int_{t_1}^{t_2} |T(t_2 - s)S(t_2 - s)F(v(s))|_\lambda\,ds \\
&\quad + \int_0^{t_2} |[T(t_2 - s)S(t_2 - s) - T(t_1 - s)S(t_1 - s)]F(v(s))|_\lambda\,ds.
\end{aligned}$$

Now write

$$\begin{aligned}
&\int_0^{t_1} |[T(t_2 - s)S(t_2 - s) - T(t_1 - s)S(t_1 - s)]F(v(s))|_\lambda\,ds \\
&\le \int_0^{t_1} |T(t_2 - s)[(-G_c)^\lambda S(t_2 - s) - (-G_c)^\lambda S(t_1 - s)](-G_c)^{-\beta}(-G_c)^\beta F(v(s))|\,ds \\
&\quad + \int_0^{t_1} |T(t_2 - s)[T(t_2 - t_1) - I](-G_c)^{-\beta}(G_c)^\lambda S(t_1 - s)(-G_c)^\beta F(v(s))|\,ds\,.
\end{aligned}$$

The equicontinuity of the family $\{Hv: v \in K\}$ follows since

$$\{(-G_c)^{-\beta}(-G_c)^\beta F(y): y \in W(x,\delta)\}$$

and

$$\{(-G_c)^{-\beta}(G_c)^{\lambda}S(t_1 - s)(-G_c)^{\beta}F(y) : y \in W(x,\delta),\ 0 \le s \le t_1 \le a\}$$

are precompact sets in X and the maps $t \to (-G_c)^{\lambda}S(t)$ and $t \to T(t)$ are continuous uniformly on compact sets of X. Also, for each $v \in K$ and $t \in [-a,a]$

$$(-G_c)^{\lambda}Hv(t) = T(t)C(t)(-G_c)^{\lambda}x + T(t)(-G_c)^{\lambda}S(t)(Bx + y)$$
$$+ \int_0^t T(t - s)(-G_c)^{-\beta}(-G_c)^{\lambda}S(t - s)(-G_c)^{\beta}F(v(s))\, ds$$

and consequently, $\{Hv(t) : v \in K \text{ and } t \in [-a,a]\}$ is precompact in X_λ. Thus by the Ascoli-Arzela Theorem $\{Hv : v \in K\}$ is precompact in X_λ and the assertions of the theorem follow from the Schauder Fixed Point Theorem.

THEOREM 3.3. *In addition to assumptions* (2.1) - (2.5), (3.1), *and* (3.2), *suppose* $(-G_c)^{-1}$ *is compact and* $0 \le \beta < \lambda$. *If* $D \subset X_\beta$ *is open and* $F: D \to X$ *is continuous, then for each* $x \in D \cap D(B)$ *and* $y \in X$ *there exist* $a > 0$ *and a continuous function* $u: [-a,a] \to X_\beta$ *satisfying* (1.2).

Proof. The proof is similar to that of the previous theorem. Let $\delta > 0$ and $N > 0$ be such that if

$$W(x,\delta) = \{z \in X : |z - x|_\beta < \delta\}$$

then $W(x,\delta) \subset D$ and $|F(z)| \le N$ for $z \in W(x,\delta)$. Define K and H as before. One shows $H: K \to K$ and is continuous. Writing

$$(-G_c)^{\beta}Hv(t) = T(t)C(t)(-G_c)^{\beta}x + T(t)(-G_c)^{\beta}S(t)(Bx + y)$$
$$+ \int_0^t T(t - s)(-G_c)^{\beta-\lambda}(-G_c)^{\lambda}S(t - s)F(v(s))\, ds$$

one observes that $\{ Hv(t): v \in K$ and $t \in [-a,a]\}$ is precompact in X_β and writing

$$|Hv(t_2) - Hv(t_1)|_\beta$$

$$\leq |T(t_2)C(t_2)x - T(t_1)C(t_1)x|_\beta + |T(t_2)S(t_2)(Bx+y) - T(t_1)S(t_1)(Bx+y)|_\beta$$

$$+ \int_{t_1}^{t_2} |T(t_2 - s)S(t_2 - s)F(v(s))|_\beta \, ds$$

$$+ \int_0^{t_1} |T(t_2 - s)[(-G_c)^\lambda S(t_2 - s) - (-G_c)^\lambda S(t_1 - s)](-G_c)^{\beta-\lambda}F(v(s))| \, ds$$

$$+ \int_0^{t_1} |T(t_2 - s)[T(t_2 - t_1) - I](-G_c)^{\beta-\lambda}(G_c)^\lambda S(t_1 - s)F(v(s))| \, ds$$

one observes that the family $\{Hv: v \in K\}$ is equicontinuous. Consequently, $H(K) \subset X$ is precompact the the assertions follow.

REMARK. In addition to the hypothesis of Theorem 3.1, 3.2, or 3.3, suppose the regularity conditions (2.6) and (2.7) hold. Then if $x \in E$ and $y \in X$, we have that u' exists and $u': [-a,a] \to X$ is continuous. If the regularity conditions (2.4) - (2.8) hold, F is continuously Frechet differentiable, $x \in D(G)$ with $Bx \in E$, and $y \in E$, then u satisfies (1.1).

The final theorem of this section gives sufficient criteria for global existence.

THEOREM 3.4. *Assume that either*

(i) the suppositions of Theorem 3.1 hold, or

(ii) the suppositions of Theorem 3.2 (3.3) hold and F *maps bounded sets of* D *into bounded sets of* X_β (X).

and the regularity conditions (2.6) - (2.7) *hold. If* $x \in E$ *and* $y \in X$ *and* u *is a solution of equation* (1.2) *noncontinuable to the right on* $[0,d]$, *then either* $d = +\infty$ *or, given any closed bounded set* $V \subset D$, *there exists a sequence* $t_k \to d^-$ *such that* $u(t_k) \notin V$. *An analogous result holds for noncontinuability to the left.*

Proof. The proofs under assumptions (i) and (ii) are similar; only assumption (ii) with Theorem 3.2 is considered. For contradiction, suppose $d < \infty$ and there exists a bounded closed set $V \subset D$ such that $u(t) \in V$ for all $t \in [0,d)$. For $0 \le t_1 < t_2 < d$ and u satisfying equation (1.2), we have

$$\begin{aligned} |u(t_2) - u(t_1)|_\lambda \le\ & |T(t_2)C(t_2)(-G_c)^\lambda x - T(t_1)C(t_1)(-G_c)^\lambda x| \\ & + |[T(t_2)(-G_c)^\lambda S(t_2) - T(t_1)(-G_c)^\lambda S(t_1)](Bx + y)| \\ & + \int_{t_1}^{t_2} |T(t_2 - s)(-G_c)^\lambda S(t_2 - s)F(u(s))|\, ds \\ & + \int_0^{t_1} |[T(t_2 - s)(-G_c)^\lambda S(t_2 - s) \\ & \qquad - T(t_1 - s)(-G_c)^\lambda S(t_1 - s)]F(u(s))|\, ds. \end{aligned}$$

Noting that $\{F(u(s)) : 0 \le s < d\} \subset F(V)$ is bounded in X_β and thus precompact in X and that $t \to T(t)(-G_c)^\lambda S(t)$ is uniformly continuous on compact sets in X we see that $\lim_{t \to d^-} u(t)$ exists with $\lim_{t \to d^-} u(t) = p \in V \subset D$. Also,

$$|u'(t_2) - u'(t_1)|$$

$$\le |T(t_2)GS(t_2)x - T(t_1)GS(t_1)x| + |T(t_2)BC(t_2)x - T(t_1)BC(t_1)x|$$

$$+ |T(t_2)C(t_2)(Bx + y) - T(t_1)C(t_1)(Bx + y)|$$

$$+ |T(t_2)BS(t_2)(Bx + y) - T(t_1)BS(t_1)(Bx + y)|$$

$$+ \int_{t_1}^{t_2} |T(t_2 - s)C(t_2 - s)F(u(s)) - T(t_2 - s)BS(t_2 - s)F(u(s))| \, ds$$

$$+ \int_0^{t_1} |[T(t_2 - s)C(t_2 - s) - T(t_1 - s)C(t_1 - s)]F(u(s))| \, ds$$

$$+ \int_0^{t_1} |[T(t_2 - s)BS(t_2 - s) - T(t_1 - s)BS(t_1 - s)]F(u(s))| \, ds.$$

Again, using the fact that $\{F(u(s)) : 0 \le s < d\}$ is precompact in X and that $t \to T(t)C(t)$ and $t \to T(t)BS(t)$ are continuous uniformly on compact sets of X, it follows that $\lim_{t \to d^-} u'(t) = q \in X$ exists. Noting that

$$p = T(d)[C(d)x + S(d)(Bx + y)] + \int_0^d T(d - s)S(d - s)F(u(s)) \, ds$$

and

$$q = T(d)[GS(d)x + C(d)(Bx + y)] - BT(d)[C(d)x + S(d)(Bx + y)]$$
$$- \int_0^d BT(d - s)S(d - s)F(u(s)) \, ds + \int_0^d T(d - s)C(d - s)F(u(s)) \, ds,$$

we see that $p \in D \cap D(B)$ and $q \in X$. Thus one can obtain a solution of the equation

$$v(t) = T(t - d)[C(t - d)p + S(t - d)(Bp + q) + \int_d^t T(t - s)S(t - s)F(v(s)) \, ds$$

for $d \le t < d^*$. One then extends u to $[0,d^*)$ by defining $u(t) = v(t)$ on $[d,d^*)$. Using the properties found in Proposition 2.1 (in particular, identities (vii) and (viii)), one can show for $t \in [d,d^*)$ that

$$u(t) = v(t) = T(t)[C(t)x + S(t)(Bx + y)] + \int_0^t T(t-s)\, S(t-s)F(u(s))\, ds,$$

contradicting the noncontinuability of u.

4. ASYMPTOTIC BEHAVIOR. In this section we will assume $T(t)$ decays exponentially as $t \to \infty$; i.e., there exists $b > 0$ such that $|T(t)| \le Me^{-bt}$ for all $t \in [0,\infty)$. For convenience, we assume $M = 1$.

THEOREM 4.1. *In addition to the suppositions of Theorem 3.2, suppose* F *maps bounded sets of* X_λ *into bounded sets of* X_β. *Also, suppose* $b < \omega_\lambda$, $x \in X_\gamma$ *for some* $\lambda < \gamma < 1$ *with* $Bx \in X_\beta$, $y \in X_\beta$ *and* u *is a solution of* (1.2) *defined and bounded on* $[0,\infty)$. *Then* $\{u(t): t \ge 0\}$ *is precompact in* X_λ. *A similar assertion holds under the hypotheses of Theorem 3.3.*

Proof. Choose $1 > \gamma > 0$ such that $\lambda < \gamma < \lambda + \beta$. Then

$$\begin{aligned}(-G_c)^\gamma u(t) &= (-G_c)^\gamma T(t)[C(t)x + S(t)(Bx + y)] \\ &\qquad + (-G_c)^\gamma \int_0^t T(t-s)S(t-s)F(u(s))\, ds \\ &= T(t)C(t)(-G_c)^\gamma x + T(t)(-G_c)^\lambda S(t)[(-G_c)^{\gamma-\lambda}(Bx + y)] \\ &\qquad + \int_0^t T(t-s)(-G_c)^\lambda S(t-s)(-G_c)^{\gamma-\lambda}F(u(s))\, ds\end{aligned}$$

and thus

$$|(-G_c)^{\gamma}u(t)| \le Ke^{(-b+\omega)t}|x|_{\gamma} + M_{\lambda}e^{(-b+\omega_{\lambda})t}|Bx + y|_{\gamma-\lambda}$$

$$+ M_{\lambda}\int_0^t e^{(-b+\omega_{\lambda})(t-s)}|F(u(s))|_{\gamma-\lambda}\, ds.$$

Consequently, $\{(-G_c)^{\gamma}u(t)|$ is bounded and $\{(-G_c)^{\lambda}u(t): t \ge 0\} = \{(-G_c)^{\lambda-\gamma}(-G_c)^{\gamma}u(t): t \ge 0\}$ is precompact in X.

THEOREM 4.2. *In addition to the hypotheses of Theorem 3.4 with* $D = X$, *suppose there exists a continuous function* $j: \mathbb{R}^+ \to \mathbb{R}^+$ *with* $j(0) = 0$ *such that* $|F(x)|_{\beta} \le j(r)|x|_{\lambda}$ *for* $x \in X_{\lambda}$ *and* $|x|_{\lambda} \le r$. *If* $x \in X_{\lambda} \cap D(B)$ *and* $y \in X$, *then there exists* $\varepsilon > 0$, $N \ge 1$, *and* $\delta > 0$ *such that if* $|x|_{\lambda} \le \varepsilon/2$ *and* $|y| \le \varepsilon/2$ *then the solution of* (1.2) *exists on* $[0,\infty)$ *and satisfies* $|u(t)|_{\lambda} \le Ne^{-\delta t}(|x|_{\lambda} + |Bx + y|)$.

Proof. Let $N = \max\{K, M_{\lambda}\}$, $\varepsilon_1 > 0$ such that $j(\varepsilon_1) < (b - \omega_{\lambda})/2$, and $\varepsilon = \varepsilon_1/N$. For $|x|_{\lambda} \le \varepsilon/2$ and $|y| \le \varepsilon/2$, let u be the solution of equation (1.2) and $[0,t^*]$ ($[0,\infty)$ if $t^* = \infty$) the maximal interval such that $|u(t)|_{\lambda} \le \varepsilon_1$ for all $0 \le t < t^*$. For $0 \le t < t^*$,

$$(-G_c)^{\lambda}u(t)$$

$$= T(t)[C(t)(-G_c)^{\lambda}x + (-G_c)^{\lambda}S(t)(Bx + y)] + \int_0^t T(t-s)(-G_c)^{\lambda}S(t-s)F(u(s))\, ds$$

and

$$|u(t)|_{\lambda}$$

$$\le e^{-bt}[Ke^{\omega t}|x|_{\lambda} + M_{\lambda}e^{\omega_{\lambda}t}|Bx + y|] + \left(\frac{b - \omega_{\lambda}}{2}\right) M_{\lambda}\int_0^t e^{-b(t-s)}e^{\omega_{\lambda}(t-s)}|u(s)|_{\lambda}\, ds.$$

Thus

$$e^{(b-\omega_\lambda)t}|u(t)|_\lambda$$

$$\leq K|x|_\lambda + M_\lambda|Bx + y| + \frac{b-\omega_\lambda}{2} M_\lambda \int_0^t e^{(b-\omega_\lambda)s}|u(s)|_\lambda \, ds$$

and Gronwall's inequality yields

$$|u(t)|_\lambda \leq N(|x|_\lambda + |Bx + y|)e^{[(b-\omega_\lambda)/2](Nt)}.$$

Thus $\delta = [(b - \omega_\lambda)/2]N$ and $t^* = \infty$ by Theorem 3.4.

5. EXAMPLES. We first consider the equation

(5.1) $$w_{tt}(x,t) + 2b(x)w(x,t) = w_{xx}(x,t) + f(w(x,t)), \quad 0 < x < \pi, \ t \in \mathbb{R}$$

$$w(0,t) = w(\pi,t) = 0, \quad t \in \mathbb{R}$$

$$w(x,0) = h(x), \quad w_t(x,0) = g(x), \quad 0 \leq x \leq \pi$$

where h and g are in $\mathcal{L}_2(0,\pi;\mathbb{R})$ and $b: [0,\pi] \to \mathbb{R}$ is continuous. Let $X = \mathcal{L}_2(0,\pi;\mathbb{R})$ with inner product $(\ ,\ )$ and define $A: D(A) \to X$ by $A\phi = \phi''$ where

$$D(A) = \{\phi \in X: \phi, \phi' \text{ are absolutely continuous,}$$
$$\phi'' \in X, \ \phi(0) = \phi(\pi) = 0\}.$$

A can be written in the form

$$A\phi = -\sum_{n=1}^{\infty} n^2(\phi,\phi_n)\phi_n$$

for $\phi \in D(A)$, where $\phi_n(x) = (2/\pi)^{1/2}\sin(nx)$. We define $B: X \to X$ by $[B\phi]x = b(x)\phi(x)$. Defining $G = A + B^2$ we have

$$[G\phi]x = \sum_{n=1}^{\infty} (-n^2 + b(x))(\phi,\phi_n)\phi_n(x)$$

and G generates the cosine family

$$[C(t)\phi](x) = \sum_{n=1}^{\infty} C_n(x,t)(\phi,\phi_n)\phi_n(x)$$

where

$$C_n(x,t) = \begin{cases} \cos(n^2 - b^2(x))^{1/2}t, & n^2 > b^2(x) \\ 1, & n^2 = b^2(x) \\ \cosh(b^2(x) - n^2)^{1/2}t, & n^2 < b^2(x). \end{cases}$$

Also,

$$[S(t)f](x) = \sum_{n=1}^{\infty} s_n(x,t)(\phi,\phi_n)\phi_n(x)$$

where

$$s_n(x,t) = \begin{cases} (n^2 - b^2(x))^{-1/2}\sin(n^2 - b^2(x))^{1/2}t, & n^2 > b^2(x) \\ t, & n^2 = b^2(x) \\ (b^2(x) - n^2)^{-1/2}\sinh(b^2(x) - n^2)^{1/2}t, & n^2 < b^2(x). \end{cases}$$

Note also that $-B$ generates the group $\{T(t): t \in \mathbb{R}\}$ on X defined by $[T(t)\phi]x = e^{-tb(x)}\phi(x)$ and $D(B) = X$. It is easily seen that properties (2.1) - (2.7) are satisfied. Property (2.8) is established by the following proposition.

PROPOSITION 5.1. *Let* A, B, $C(t)$, $S(t)$, $T(t)$ *be as above. Then* (2.8) *is satisfied, i.e., for* $\phi \in X$, $\int_r^s T(u)S(u)\phi\, du \in D(A)$ *and*

$$A\int_r^s T(u)S(u)\phi\, du = T(s)C(s)\phi - T(r)C(r)\phi + BT(s)S(s)\phi - BT(r)S(r)\phi.$$

Proof. Using the fact that $C(t)T(s) = T(s)C(t)$ for all $t,s \in \mathbb{R}$ and identity (viii) in Proposition (2.1), we have

$$j(t) \overset{\text{def}}{\equiv} C(t) \int_r^s T(u)S(u)\phi \, du = \frac{1}{2} \int_r^s T(u)(S(u+t) + S(u-t))\phi \, du.$$

Thus

$$j'(t) = \frac{1}{2} \int_r^s T(u)(C(u+t) - C(u-t))\phi \, du$$

$$= \frac{1}{2} \int_{r+t}^{s+t} T(u-t)C(u)\phi \, du - \frac{1}{2} \int_{r-t}^{s-t} T(u+t)C(u)\phi \, du$$

and

$$j''(t) = \frac{1}{2} [T(s)C(s+t)\phi - T(r)C(r)\phi] + \frac{1}{2} \int_{r+t}^{s+t} BT(u-t)C(u)\phi \, du$$

$$+ \frac{1}{2} [T(s)C(s-t)\phi - T(r)C(r-t)\phi] + \frac{1}{2} \int_{r-t}^{s-t} BT(u+t)C(u)\phi \, du.$$

Integrating by parts,

$$(A + B^2) \int_r^s T(u)S(u)\phi du = C''(0) \int_r^s T(u)S(u)\phi du$$

$$= T(s)C(s)\phi - T(r)C(r)\phi + \int_r^s BT(u)C(u)\phi \, du$$

$$= T(s)C(s)\phi - T(r)C(r)\phi + BT(s)S(s)\phi - BT(r)S(r)\phi$$

$$+ \int_r^s B^2 T(u)C(u)\phi \, du$$

from which (2.8) follows.

As noted in the comments preceding condition (3.1), condition (3.1) is satisfied with $\lambda = 1/2$. Also, if c is such that $b^2(x) - c^2 \le 0$

for all $x \in [0,\pi]$, we see that G_c^{-1} exists, satisfies

$$[G_c^{-1}\phi](x) = \sum_{n=1}^{\infty} (-n^2 + b^2(x) - c^2)^{-1}(\phi,\phi_n)\phi_n(x),$$

and is compact. Furthermore,

$$[(-G_c)^{1/2}\phi]x = \sum_{n=1}^{\infty} (n^2 - b^2(x) + c^2)^{1/2}(\phi,\phi_n)\phi_n(x).$$

Since

$$D((-G_c)^{1/2}) = \{\phi \in X: \sum_{n=1}^{\infty} (n^2 - b(x) + c^2)(\phi,\phi_n)^2 < \infty\}$$

$$= \{\phi \in X: \sum_{n=1}^{\infty} n^2(\phi,\phi_n)^2 < \infty\},$$

we have from Travis and Webb [7] that

$$D((-G_c)^{1/2}) = \{\phi \in X: \phi \text{ is absolutely continuous, } \phi' \in X, \text{ and } \phi(0) = \phi(\pi) = 0\}.$$

Noting that for $\phi \in D((-G_c)^{1/2})$

$$|\phi'| = \int_0^{\pi} \left[\sum_{n=1}^{\infty} (\phi',\phi_n)\phi_n(x)\right]^2 dx$$

$$= \int_0^{\pi} \left[\sum_{n=1}^{\infty} n\left(\frac{2}{\pi}\right)^{1/2}(\phi,\phi_n)\cos(nx)\right]^2 dx$$

$$= \sum_{n=1}^{\infty} n^2(\phi,\phi_n)^2$$

and

$$|\phi|_{1/2} = |-G_c^{1/2}\phi|$$

$$= \int_0^\pi \left[\sum_{n=1}^{\infty} (n^2 - b^2(x) + c^2)^{1/2}(\phi,\phi_n)\phi_n(x)\right]^2 dx$$

$$= \sum_{n=1}^{\infty} (n^2 - b^2(x) + c^2)(\phi,\phi_n)^2$$

it follows that there exists $K > 0$ such that for all $\phi \in D((-G_c)^{1/2})$

$$K|\phi'| \geq |\phi|_{1/2} \geq |\phi'|.$$

If $f(w) = -aw - bw^3$, $a,b > 0$ or $f(w) = \sin w$ then equation (5.1) is the Klein-Gordon or Sine-Gordon equation respectively and $[F(\phi)]x = f(\phi(x))$ satisfies the conditions of Theorem 3.2 with $F: X_{1/2} \to X_{1/2}$. For example, to see $F: X_{1/2} \to X_{1/2}$ defined by $[F(\phi)]x = -a\phi(x) - b\phi^3(x)$ is continuous, observe for $\phi,\psi \in X_{1/2}$

$$|F\phi - F\psi|_{1/2} \leq K|[F\phi]' - [F\psi]'|$$

$$\leq K|a(\phi' - \psi') + 3b(\psi^2\psi' - \phi^2\phi')|$$

$$\leq aK|\phi' - \psi'| + 3bK(|\psi^2\psi' - \psi^2\phi'| + |\phi'(\psi^2 - \phi^2)|)$$

Supp Suppose $b(x) \geq 0$ for all $x \in [0,\pi]$ and let $b_m = \min\{b(x): 0 \leq x \leq \pi\}$ and $b_M = \max\{b(x): 0 \leq x \leq \pi\}$. Then $|T(t)| \leq e^{-b_m t}$ and there exists $M_{1/2} > 0$ such that $|(-G_c)^{1/2}S(t)| \leq M_{1/2}e^{w_{1/2}t}$ where $w_{1/2} = 0$ if $b_M \leq 1$ and $w_{1/2} = (b_M^2 - 1)^{1/2}$ if $b_M > 1$. Consequently, the results of section 4 apply provided $0 < b_m < b_M \leq 1$ or $b_M > 1$ and $b_m > (b_M^2 - 1)^{1/2}$.

As another example, consider

$$(5.2) \qquad w_{tt}(x,t) + 2w_{xt}(x,t) = w_{xx}(x,t) + f(w(x,t)), \quad 0 < x < \pi, \ t \in \mathbb{R}$$

$$w(x,0) = h(x), \quad w_t(x,0) = g(x), \quad 0 \le x \le \pi$$

$$w(0,t) = w(\pi,t), \quad w_x(0,t) = w_x(\pi,t), \quad t \in \mathbb{R}$$

where $h,g \in \mathcal{L}_2(0,\pi;\mathbb{R})$. Again let $X = \mathcal{L}_2(0,\pi;\mathbb{R})$ with inner product $(\ ,\ )$ and define $A\colon D(A) \to X$ by $A\phi = \phi''$ where

$$D(A) = \{\phi \in X\colon \phi,\ \phi' \text{ are absolutely continuous,}$$
$$\phi'' \in X,\ \phi(0) = \phi(\pi),\ \phi'(0) = \phi'(\pi)\}.$$

A can be written in the form

$$A\phi = \sum_{n=1}^{\infty} -4n^2[(\phi,\phi_n)\phi_n + (\phi,\psi_n)\psi_n]$$

where $\phi_n(x) = (2/\pi)^{1/2}\sin(2nx)$, $\psi_n(x) = (2/\pi)^{1/2}\cos(2nx)$, and $\phi = a + \Sigma_{n=1}^{\infty} (\phi,\phi_n)\phi_n + (\phi,\psi_n)\psi_n$, $a = (1/\pi) \int_0^{\pi} \phi(x)\,dx$. We define $B\colon D(B) \to X$ by $B\phi = \phi'$ where

$$D(B) = \{\phi \in X\colon \phi' \in X,\ \phi(0) = \phi(\pi)\}.$$

The group $\{T(t)\colon t \in \mathbb{R}\}$ generated by $-B$ is defined by

$$[T(t)\phi]x = a + \sum_{n=1}^{\infty} (\phi,\phi_n)\phi_n(x - t) + (\phi,\psi_n)\psi_n(x - t)$$

and $G = A + B^2 = 2A$ generates the cosine family

$$[C(t)\phi]x = a + \sum_{n=1}^{\infty} \cos(2\sqrt{2}\,nt)[(\phi,\phi_n)\phi_n(x) + (\phi,\psi_n)\psi_n(x)].$$

Conditions (2.1) - (2.7) are satisfied and condition (2.8) is verified in the manner indicated by the proof of Proposition 5.1. Also, G^{-1} exists as a compact operator on X and $(-G_c)^{1/2}$ exists with

$$(-G_c)^{1/2}\phi = \sum_{n=1}^{\infty} 2\sqrt{2}\, n[(\phi,\phi_n)\phi_n + (\phi,\psi_n)\psi_n].$$

The existence results of section 3 apply to example (5.2); however, note that since $T(t)$ is of type $b = 0$ and $C(t)$ is of type $w = 0$, the asymptotic results of section 4 do not apply.

The abstract theory also applies to the equation

$$(5.3)\qquad \begin{aligned} &w_{tt} + 2b(x)w_t = -w_{xxxx} + f(w,w_x), \quad 0 < x < \pi, \quad t \in \mathbb{R} \\ &w(0,t) = w(\pi,t) = w_{xx}(0,t) = w_{xx}(\pi,t) = 0, \quad t \in \mathbb{R} \\ &w(x,0) = g(x), \quad w_t(x,0) = h(x), \quad 0 \le x \le \pi . \end{aligned}$$

As before let $X = \mathcal{L}_2(0,\pi;\mathbb{R})$, $A\phi = -\phi''''$ with

$$D(A) = \{\phi \in X\colon \phi, \phi', \phi'', \phi''' \text{ are absolutely continuous, } \phi'''' \in X,\ \phi(0) = \phi(\pi) = \phi''(0) = \phi''(\pi) = 0\}.$$

In this case, $[F\phi]x = f(\phi(x),\phi'(x))$ with appropriate conditions on f satisfies $F\colon X_{1/4} \to X$ continuous and consequently Theorem 3.3 applies.

REMARK. The techniques also apply to

$$(5.4)\qquad w_{tt} - 2w_{xxt} = -w_{xxxx} + f(w,w_x), \quad 0 < x < \pi, \quad t \ge 0$$

with the side conditions of (5.3). Here $B\phi\colon -\phi''$ with

$$D(B) = \{\phi \in X\colon \phi,\phi' \text{ are absolutely continuous, } \phi'' \in X,\ \phi(0) = \phi(\pi) = 0\}.$$

Thus $-B$ generates an analytic semigroup. Since $A + B^2 = 0$, $C(t)x = x$, $S(t)x = tx$, and equation (1.2) has the form

$$u(t) = T(t)[x + t(Bx + y)] + \int_0^t (t - s)T(t - s)F(u(s))\, ds.$$

REFERENCES

1. H. Fattorini, Ordinary differential equations in linear topological spaces, I, *J. Differential Equations* 5(1968), 73-105.

2. S. Rankin, A remark on cosine families, *Proc. Amer. Math. Soc.* 79(1980), 376-378.

3. J. Rauch, Qualitative behavior of dissipative wave equations on bounded domains, *Arch. Rat. Mech. Anal.* 62(1976), 77-85.

4. C. Travis and G. Webb, Cosine families and abstract nonlinear second order differential equations, *Acta Math. Acad. Sci. Hung.* 32(1978), 75-96.

5. ______, Perturbation of strongly continuous cosine family generators, *Colloq. Math.*, to appear.

6. ______, An abstract second order semilinear Volterra integrodifferential, *Siam Journal of Math. Anal.*, 10(1979), 412-424.

7. ______, Partial differential equations with deviation arguments in the time variable, *J. Math. Anal. Appl.* 56(1976), 397-409.

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Consider the abstract differential equation

$$(1)\quad u''(t) + 2Bu'(t) = Au(t) + F(u(t)),\ t \in \mathbb{R},\ u(0) = x,\ u'(0) = y$$

continued

20. Abstract (continued)

where A and B are densely defined linear operators and F is possibly nonlinear and unbounded. Assuming that $A + B^2$ generates a cosine family $C(t)$ and $-B$ generates a group $T(t)$, there is a variation of constants formula for (1); namely

$$(2)\quad u(t) = T(t)[C(t)x + S(t)(Bx + y)] + \int_0^t T(t - s)S(t - s)F(u(s))\,ds,$$

where $S(t)$ is the sine family associated with $C(t)$. The motivating examples include $w_{tt} + 2b(x)w_t = w_{xx} + f(w,w_x,w_t)$ and $w_{tt} + 2w_{tx} = w_{xx} + f(w,w_x,w_t)$, for $0 < x < \pi$, $t \in \mathbb{R}$, $w(x,0) = h(x)$, $w_t(x,0) = g(x)$, and various boundary conditions. We examine the existence of mild solutions and the asymptotic behavior when there is a damping effect introduced by the $2Bu'(t)$ term.

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